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A Method for The Construction of Pfaffian Systems with Finite Monodromy

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Introduction. Let Ω be a square-matrix-valued meromorphic 1-form on a complex manifold M such that $d\Omega + \Omega\wedge\Omega = 0$. For an unknown square-matrix-valued function F, the differential equation

$$dF = F\Omega$$

is called a *Pfaffian system.*

There is an important class of Pfaffian systems, called the class of Pfaffian systems *of Fuchsian type* (Deligne [2]). Some important differential equations such as Gauss' hypergeometric differential equations can be expressed as Pfaffian systems of Fuchsian type.

For a Pfaffian system, its monodromy group is defined. It is however a difficult problem in general to compute the monodromy group of a given Pfaffian system.

Hence it is also a difficult problem to determine if the monodromy group is finite or not. As for Gauss' hypergeometric differential equations, this problem was solved by Schwarz [9].

In this paper, we discuss the converse problem. That is, we give a general method for the construction of a wide class of Pfaffian systems, including those of Fuchsian type, with given finite monodromy groups.

1. Finite Galois coverings of complex manifolds. Let M be a complex manifold. (The connectedness is always assumed.) We fix M once for all. A finite (branched) covering of M is , by definition, a finite proper holomorphic mapping

$$\pi : X \longrightarrow M$$

of an irreducible normal complex space X onto M.

Let $\pi : X \longrightarrow M$ and $\mu : Y \longrightarrow M$ be two finite coverings of M. A morphism (resp. an isomorphism) of π to μ is, by definition, a holomorphic (resp. biholomorphic) mapping φ of X onto Y such that $\mu \cdot \varphi = \pi$. When there is a morphism (resp. an isomorphism) of π to μ, we denote

$$\pi \geqslant \mu \quad \text{or} \quad \mu \leqslant \pi \qquad (\text{resp. } \pi \simeq \mu).$$

The set $\mathrm{Aut}(\pi)$ of all automorphisms of π forms a group under compositions and is called the automorphism group of π, which acts on every fiber of π.

A finite covering $\pi : X \longrightarrow M$ is called a finite Galois covering if $\mathrm{Aut}(\pi)$ acts transitively on every fiber of π. In this case, the quotient complex space $X/\mathrm{Aut}(\pi)$ (see Cartan [1]) is naturally biholomorphic to M.

Let $\pi : X \longrightarrow M$ be a finite covering. We put

$$R_\pi = \left\{ p \in X \;\middle|\; \pi \text{ is not biholomorphic around } p \right\}$$

and
$$B_\pi = \pi(R_\pi).$$

Then R_π and B_π are hypersurfaces (i.e., codimension 1 at every point) of X and M, respectively (see Fischer [3]), called the <u>ramification locus</u> and the <u>branch locus of</u> π, respectively.

Let B be a hypersurface of M. A finite covering $\pi: X \longrightarrow M$ is said to <u>branch at most at</u> B if B contains the branch locus B_π of π. In this case, the restriction π' of π to $X - \pi^{-1}(B)$

$$\pi': X - \pi^{-1}(B) \longrightarrow M - B \qquad \cdots\cdots (1)$$

is a usual (unbranched) covering. Its mapping degree is (independent of B and is) called the <u>degree of</u> π and is denoted by $\deg \pi$.

We have easily

<u>Lemma 1.1.</u> Let B be a hypersurface of M. Let π and μ be finite coverings of M which branch at most at B. Let π' and μ' be the restrictions of π and μ, respectively, as in (1). Then (i) $\pi \geqslant \mu$ if and only if $\pi' \geqslant \mu'$ and (ii) $\pi \simeq \mu$ if and only if $\pi' \simeq \mu'$.

<u>Lemma 1.2.</u> Under the same notations as in Lemma 1.1, (i) $\mathrm{Aut}(\pi)$ is naturally isomorphic to $\mathrm{Aut}(\pi')$ and (ii) π is a finite Galois covering if and only if π' is a finite Galois covering.

<u>Corollary 1.3.</u> $\#\mathrm{Aut}(\pi) \leqslant \deg \pi$, where $\#G$ means the

order of the group G. The equality holds if and only if π is a finite Galois covering.

The following theorem is deep.

Theorem 1.4 (Grauert-Remmert [4], see also Grothendieck-Raynaud [5]). Let B be a hypersurface of a complex manifold M. Let $\pi': X' \longrightarrow M - B$ be a finite unbranched covering. Then there exists a unique (up to isomorphisms) finite covering π: $X \longrightarrow M$ branching at most at B whose restriction to $X - \pi^{-1}(B)$ is isomorphic to π'.

Let o be a fixed point of $M - B$. We denote by $\pi_1(M - B, o)$ the fundamental group of $M - B$ with the reference point o.

By Theorem 1.4,

Theorem 1.5. Let B be a hypersurface of a complex manifold M. Then there exists a one-to-one correspondence $\pi \longmapsto H = H(\pi)$ between the set of all isomorphism classes of finite Galois coverings π of M branching at most at B and the set of all normal subgroups H of finite index of $\pi_1(M - B, o)$, which satisfies the following two conditions: (i) $\pi \geqslant \mu$ if and only if $H(\pi) \subset H(\mu)$ and (ii) $\mathrm{Aut}(\pi)$ is naturally isomorphic to $\pi_1(M - B, o)/H(\pi)$.

2. Pfaffian systems of meromorphic type with finite monodromy.

Let $\Omega = (\omega_{jk})$ be an $(m \times m)$-matrix-valued meromorphic 1-form on a complex manifold M which satisfies the integrability condition:

$$d\Omega + \Omega \wedge \Omega = 0 . \qquad \cdots\cdot (2)$$

Let B_{jk} be the polar set of ω_{jk}. We put

$$B = \bigcup_{j,k} B_{jk}$$

and call it the <u>polar set of</u> Ω. B is a hypersurface of M. We put $M' = M - B$ and denote by $\widetilde{M}'$ the universal covering space of M'.

A theorem of Frobenius asserts that the Pfaffian system

$$dF = F\Omega \qquad \cdots\cdot (3)$$

has a solution F such that (i) F is an $(m \times m)$-matrix-valued holomorphic function on $\widetilde{M}'$, (ii) the determinant $\det F$ of F is nowhere vanishing and (iii) any other solution of (3) can be written as AF, where A is a constant $(m \times m)$-matrix.

We call such an F a <u>fundamental solution of</u> (3).

Let $o \in M' = M - B$ be a fixed point. The fundamental group $\pi_1(M', o)$ naturally acts on $\widetilde{M}'$. Let F be a fundamental solution of (3). For each element $\gamma \in \pi_1(M', o)$, we put $\gamma^* F = F \cdot \gamma$. Then, by (3),

$$d(\gamma^* F) = \gamma^* dF = \gamma^*(F\Omega) = (\gamma^* F)\Omega.$$

Hence we may write

$$\gamma^* F = R(\gamma)F \qquad \text{for } \gamma \in \pi_1(M', o), \qquad \cdots\cdot (4)$$

where

$$R : \gamma \in \pi_1(M', o) \longmapsto R(\gamma) \in GL(m, \mathbb{C})$$

is a homomorphism, called the monodromy representation of the Pfaffian system (3). Its image $G = R(\pi_1(M', o))$ is called the monodromy group of (3).

Assume now that the monodromy group G is a finite subgroup of $GL(m,\mathbb{C})$. In this case, (3) is called a Pfaffian system with finite monodromy. The kernel $\mathrm{Ker}(R)$ of R is a normal subgroup of finite index of $\pi_1(M', o)$. By Theorem 1.5, there corresponds a unque (up to isomorphisms) finite Galois covering

$$\pi : X \longrightarrow M \qquad \cdots\cdots (5)$$

such that $\mathrm{Aut}(\pi) \simeq G$ naturally. A fundamental solution F can be regarded as a holomorphic function on $X' = \pi^{-1}(M') = X - \pi^{-1}(B)$. In general, F has essential singularity along $\pi^{-1}(B)$.

The Pfaffian system (3) is said to be of meromorphic type with finite monodromy if (i) its monodromy group G is a finite subgroup of $GL(m,\mathbb{C})$ and (ii) F can be extended to a meromorphic function on the normal complex space X in (5).

Proposition 2.1. Every Pfaffian system of Fuchsian type with finite monodromy is of meromorphic type with finite monodromy.

Proof. If the Pfaffian system (3) is of Fuchsian type, then (see Deligne [2]), around a generic point q of B, Ω can be written as

$$\Omega = A_1(w)dw_1/w_1 + A_2(w)dw_2 + \cdots + A_n(w)dw_n,$$

where (i) $(w_1, w_2, \cdots, w_n)$ is a local coordinate system in M

around q such that $q = (0,0,\cdots,0)$ and $B = \{w_1 = 0\}$ (around q) and (ii) $A_j(w) = A_j(w_1,w_2,\cdots,w_n)$ are $(m\times m)$-matrix-valued holomorphic functions.

By Yoshida-Takano [11], a fundamental solution F of (3) can be written as

$$F(w) = (\exp C\log w_1)(\exp N\log w_1)G(w)$$

around q, where C is a constant $(m\times m)$-matrix, N is a diagonal $(m\times m)$-matrix with integral coefficients and $G(w)$ is an $(m\times m)$-matrix-valued holomorphic function with $\det G(w)$ nowhere vanishing.

If (3) is of finite monodromy, then C must be diagonalizable and every eigenvalue of C must be a rational number. Thus $F(w)$ can be meromorphically extended to an open neighborhood of $\pi^{-1}(q)$ in X.

By Levi's extension theorem (see Fischer [3]), F can be meromorphically extended to X.

q.e.d.

3. Construction. Put $s = m^2$ and let $Y = \mathbb{P}^s$ be the s-dimensional complex projective space. Y is the disjoint union of $\mathbb{C}^s$ and H_∞, where H_∞ is the hyperplane at infinity. We identify $\mathbb{C}^s$ with the set of all complex $(m\times m)$-matrices.

$GL(m,\mathbb{C})$ acts on $\mathbb{C}^s$ as the product of matrices:

$$(A, y)\in GL(m,\mathbb{C})\times\mathbb{C}^s \longmapsto Ay\in\mathbb{C}^s.$$

This action can be naturally extended to that on Y by defining

$$(A,\ (0:y)) \in GL(m,\mathbb{C}) \times H_\infty \longmapsto (0:Ay) \in H_\infty .$$

Put
$$\Delta = \left\{ y \in \mathbb{C}^s \;\middle|\; \det y = 0 \right\} \bigcup H_\infty . \qquad \cdots\cdots (6)$$

Then Δ is a hypersurface of Y which is invariant under the action of $GL(m,\mathbb{C})$.

The $(m \times m)$-matrix-valued meromorphic 1-form

$$\Xi = y^{-1} dy \qquad \cdots\cdots (7)$$

on Y is clearly $GL(m,\mathbb{C})$-invariant.

Now, suppose that the Pfaffian system (3) on a complex manifold M is given and is of meromorphic type with finite monodromy. Then the fundamental solution F of (3) can be regarded as a meromorphic mapping

$$F:\ X \longrightarrow Y = \mathbb{P}^s,$$

where $\pi : X \longrightarrow M$ is the finite Galois covering in (5). (4) means that the meromorphic mapping F is equivariant under the actions of $\mathrm{Aut}(\pi)$ on X and of the monodromy group G on Y. Hence a meromorphic mapping

$$f:\ M \longrightarrow N = Y/G \qquad \cdots\cdots (8)$$

is indeced from F and makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mu} \\ M & \xrightarrow{f} & N = Y/G \end{array}$$

commutative. (μ is the natural projection.)

In general,

<u>Definition 3.1.</u> Let G be a finite subgroup of $GL(m,\mathbb{C})$. A meromorphic mapping $g: M \longrightarrow N = Y/G$ is said to be <u>G-primitive</u> if (i) there is a hypersurface B of M such that g is holomorphic on $M - B$ and $g(M-B) \cap \mu(\Delta) = \phi$, where Δ is defined in (6) and (ii) there is no decomposition of g as follows: $g = h \cdot \nu$. Here, $h: M \longrightarrow Y/H$ is a meromorphic mapping and $\nu : Y/H \longrightarrow N = Y/G$ is the natural projection, where H is a proper subgroup of G.

<u>Remark 3.2.</u> In Namba [8] , a meromorphic mapping $g: M \longrightarrow N = Y/G$ was said to be <u>G-indecomposable</u> if g satisfies (i)' $g(M) \not\subset \mu(\text{Fix } G)$ and (ii) in Definition 3.1. Here, $\text{Fix } G = \bigcup \text{Fix } A$, where $\text{Fix } A$ is the fixed point set of A and the union runs over all elements A of G with $A \neq 1$. It is clear that a G-primitive meromorphic mapping is G-indecomposable.

We can easily show (see Namba [8]) that the meromorphic mapping $f: M \longrightarrow N = Y/G$ in (8) is G-primitive.

The $(m \times m)$-matrix-valued 1-form Ξ on Y in (7) is $GL(m,\mathbb{C})$-invariant. So, it is G-invariant. Hence it can be regarded as an $(m \times m)$-matrix-valued rational 1-form on $L/\mathbb{C}$, (see Iitaka [6]), where $L = \mathbb{C}(N)$ is the field of meromorphic functions on $N = Y/G$.

In fact, if we take algebraically independent elements

$$u_1, u_2, \cdots, u_s \qquad (s = m^2)$$

in L, then $L/\mathbb{C}(u_1, u_2, \cdots, u_s)$ is a finite extension. Put $K = \mathbb{C}(Y)$. Then K/L is also a finite extension. By Iitaka [6] ,

Ξ can be written as

$$\Xi = \Xi_1 du_1 + \Xi_2 du_2 + \cdots + \Xi_s du_s, \qquad \cdots (9)$$

where Ξ_j are $(m \times m)$-matrix-valued meromorphic functions on Y. For each element A of G, we have

$$\Xi = A^*\Xi = A^*\Xi_1 du_1 + A^*\Xi_2 du_2 + \cdots + A^*\Xi_s du_s.$$

Since $du_1, du_2, \cdots, du_s$ are linearly independent over K, we have $A^*\Xi_j = \Xi_j$ $(1 \leqslant j \leqslant s)$. Hence Ξ_j are $(m \times m)$-matrix-valued meromorphic functions on $N = Y/G$. Thus Ξ can be regarded as an $(m \times m)$-matrix-valued rational 1-form on $L/\mathbb{C}$.

<u>Example 3.3.</u> Put $m = 2$ and $G = \{1, A\}$, where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We put

$$y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$

and

$$p = -y_{11} - y_{21}, \qquad q = y_{11}y_{21},$$

$$r = -y_{12} - y_{22}, \qquad s = y_{12}y_{22},$$

$$t = y_{11}y_{12} + y_{21}y_{22}, \qquad u = y_{11}y_{22} + y_{12}y_{21}.$$

Then p, q, r, s, t and u generate the ring $\mathbb{C}[y_{11}, y_{21}, y_{12}, y_{22}]^G$ of G-invariants and have the following two relations:

$$t + u = pr,$$

$$tu = p^2 s + r^2 q - 4qs.$$

In this case, $\Xi = y^{-1}dy$ is written as

$$\Xi = \begin{pmatrix} \dfrac{-udp+rdq}{-pu+2qr} & \dfrac{2sdr-rds}{-ru+2ps} \\ \\ \dfrac{2qdp-pdq}{-pu+2qr} & \dfrac{-udr+pds}{-ru+2ps} \end{pmatrix}$$

which is a (2×2)-matrix-valued rational 1-form on $L/\mathbb{C}$, where L is the quotient field of $\mathbb{C}[y_{11}, y_{21}, y_{12}, y_{22}]^G$.

Now, let

$$\begin{tikzcd} M_o \arrow[r, "f_o"] \arrow[d, "\rho"'] & N \arrow[d, "id"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

be the resolution of indeterminacy of the meromorphic mapping f in (8), where id is the identity mapping and ρ is a proper modification (see Ueno [10]).

We operate f_o^* on Ξ . Then, by (9), we have

$$f_o^*\Xi = (f_o^*\Xi_1)d(f_o^*u_1) + (f_o^*\Xi_2)d(f_o^*u_2) + \cdots + (f_o^*\Xi_s)d(f_o^*u_s).$$

This is well defined by the condition (i) in Definition 3.1 and is an $(m\times m)$-matrix-valued meromorphic 1-form on M_o. Moreover, we can easily see that the original meromorphic 1-form Ω on M is recovered by the relation

$$f_o^*\Xi = \rho^*\Omega. \qquad \cdots\cdot(10)$$

(Note that $\rho : M_o - \rho^{-1}(S) \longrightarrow M - S$ is biholomorphic, where S is the points of indeterminacy of f.)

Conversely, if $f: M \longrightarrow N = Y/G$ is a G-primitive meromorphic mapping for a given finite subgroup G of $GL(m,\mathbb{C})$, then we can define an $(m\times m)$-matrix-valued meromorphic 1-form Ω on M by (10). Ω satisfies the integrability condition (2), for Ξ satisfies it. Let $M_o \times_N Y$ be the fiber product of M_o and Y over N. Then $M_o \times_N Y$ is irreducible (see Namba [8]). Let

$$\alpha : X_o \longrightarrow M_o \times_N Y$$

be its normalization and put $\pi_o = \pi' \cdot \alpha$, where

$$\pi' : M_o \times_N Y \longrightarrow M_o$$

is the natural projection. Then

$$\pi_o : X_o \longrightarrow M_o$$

is a finite Galois covering with $\mathrm{Aut}(\pi_o) \simeq G$ naturally. By Theorem 1.4, π_o induces a finite Galois covering

$$\pi : X \longrightarrow M$$

with $\mathrm{Aut}(\pi) \simeq G$ naturally. Moreover, there is a commutative diagram:

$$\begin{tikzcd} X_o \arrow[r, "\eta"] \arrow[d, "\pi_o"'] & X \arrow[d, "\pi"] \\ M_o \arrow[r, "\rho"'] & M \end{tikzcd}$$

where η is a proper modification.

Now, look at the following commutative diagram:

$$\begin{tikzcd} X_o \arrow[r, "\alpha"] \arrow[d, "\pi_o"'] & M_o \times_N Y \arrow[r, "\beta"] \arrow[d, "\pi'"'] & Y \arrow[d, "\mu"'] \\ M_o \arrow[r, "id"'] & M_o \arrow[r, "f"'] & N = Y/G \end{tikzcd}$$

Here, id is the identity mapping and β is the natural projection. The holomorphic mapping

$$F_o = \beta \cdot \alpha : X_o \longrightarrow Y$$

infuces a meromorphic mapping

$$F : X \longrightarrow Y = \mathbb{P}^s \qquad (s = m^2)$$

such that $F \cdot \eta = F_o$.

It is now easy to see that F gives a fundamental solution of the Pfaffian system

$$dF = F\Omega ,$$

which is of meromorphic type with the monodromy group G. Thus we conclude

Theorem 3.4. Let M be a complex manifold and G be a finite subgroup of $GL(m,\mathbb{C})$. Then every Pfaffian system on M of meromorphic type with the monodromy group G can be obtained by Ω in (10) for a G-primitive meromorphic mapping $f: M \longrightarrow N = \mathbb{P}^s/G \quad (s = m^2)$.

Remark 3.5. (i) An idea similar to our method can be found in Klein [7]. (ii) Our method can also be applied to Pfaffian systems with discrete monodromy groups. (iii) It is not easy in general to check the condition (ii) in Definition 3.1 for a given meromorphic mapping $f: M \longrightarrow Y/G$. If a meromorphic mapping $f: M \longrightarrow Y/G$ satisfies only the condition (i) in Definition 3.1, then, by the same method, we still have a Pfaffian system on M of meromorphic type with finite monodromy, whose monodromy group is however a subgroup of G. (iv) If

$f_o: M_o \longrightarrow Y/G$ is surjective with connected fibers, then we can show that $f: M \longrightarrow Y/G$ is G-primitive (see Namba [8]).

<u>Example 3.6.</u> We take the same group $G = \{1, A\}$ as in Example 3.3 and use the same notations. A meromorphic mapping $f: M \longrightarrow N = \mathbb{P}^4/G$ can be written as

$$f = (f_1, f_2, f_3, f_4, f_5, f_6) = (p, q, r, s, t, u),$$

where f_j $(1 \leqslant j \leqslant 6)$ are meromorphic functions on M such that

$$f_5 + f_6 = f_1 f_3 \qquad \text{and}$$

$$f_5 f_6 = f_1^2 f_4 + f_3^2 f_2 - 4 f_2 f_4.$$

In this case, f is G-primitive if and only if (i) $f_6^2 - 4f_2 f_4 \neq 0$ and (ii) one of the following quadratic equations does not have a solution in $\mathbb{C}(M)$, the field of meromorphic functions on M:

$$x^2 + f_1 x + f_2 = 0,$$

$$x^2 + f_3 x + f_4 = 0.$$

Suppose that f is G-primitive. Then the Pfaffian system $dF = F\Omega$ is of meromorphic type with the monodromy group $G = \{1, A\}$, where

$$\Omega = \begin{pmatrix} \dfrac{-f_6 df_1 + f_3 df_2}{-f_1 f_6 + 2f_2 f_3} & \dfrac{2f_4 df_3 - f_3 df_4}{-f_3 f_6 + 2f_1 f_4} \\[4ex] \dfrac{2f_2 df_1 - f_1 df_2}{-f_1 f_6 + 2f_2 f_3} & \dfrac{-f_6 df_3 + f_1 df_4}{-f_3 f_6 + 2f_1 f_4} \end{pmatrix}.$$

Conversely, by Theorem 3.4, every Pfaffian system on M

of meromorphic type with the monodromy group $G = \{1, A\}$ can be obtained in this way.

<u>Example 3.7.</u> Put $G = \{1, A, B, AB\}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We put

$$y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$

and

$$u_1 = y_{11}^2,\ u_2 = y_{11}y_{12},\ u_3 = y_{12}^2,$$

$$v_1 = y_{21}^2,\ v_2 = y_{21}y_{22},\ v_3 = y_{22}^2.$$

Then u_1, u_2, u_3, v_1, v_2 and v_3 generate the ring $\mathbb{C}[y_{11}, y_{12}, y_{21}, y_{22}]^G$ of G-invariants and have the following two relations:

$$u_1u_3 = u_2^2, \quad v_1v_3 = v_2^2.$$

Hence a meromorphic mapping $f: M \longrightarrow N = \mathbb{P}^4/G$ can be written as

$$f = (f_1, f_2, f_3, g_1, g_2, g_3) = (u_1, u_2, u_3, v_1, v_2, v_3),$$

where f_j and g_j $(1 \leqslant j \leqslant 3)$ are meromorphic functions on M such that

$$f_1f_3 = f_2^2 \quad \text{and} \quad g_1g_3 = g_2^2.$$

In this case, f is G-primitive if and only if (i) $f_1g_3 \neq f_3g_1$ and (ii) none of the following two equations has a solution in $\mathbb{C}(M)$:

$$x^2 - f_1 = 0, \quad x^2 - g_1 = 0.$$

Suppose that f is G-primitive. Then the Pfaffian system $dF = F\Omega$ is of meromorphic type with the monodromy group $G = \{1, A, B, AB\}$, where

$$\Omega = \frac{1}{f_1g_3 - f_3g_1}\begin{pmatrix} \dfrac{(f_1g_3+f_2g_2)df_1}{2f_1} - \dfrac{(f_2g_2+f_3g_1)dg_1}{2g_1} & \dfrac{(f_2g_3+f_3g_2)df_3}{2f_3} - \dfrac{(f_2g_3+f_3g_2)dg_3}{2g_3} \\ \dfrac{(f_1g_2+f_2g_1)dg_1}{2g_1} - \dfrac{(f_1g_2+f_2g_1)df_1}{2f_1} & \dfrac{(f_1g_3+f_2g_2)dg_3}{2g_3} - \dfrac{(f_2g_2+f_3g_1)df_3}{2f_3} \end{pmatrix}.$$

Conversely, every Pfaffian system on M of meromorphic type with the monodromy group $G = \{1, A, B, AB\}$ can be obtained in this way.

<u>4. Finite projective monodromy groups.</u> Next, let $PGL(m,\mathbb{C})$ be the projective linear group and

$$1 \longrightarrow \mathbb{C}^* \longrightarrow GL(m,\mathbb{C}) \xrightarrow{\lambda} PGL(m,\mathbb{C}) \longrightarrow 1$$

the the natural exact sequence, where $\mathbb{C}^* = \mathbb{C} - \{0\}$.

For a Pfaffian system (3), let R be its monodromy representation. The homomorphism

$$\hat{R} = \lambda\cdot R\colon \pi_1(M - B,\ o) \longrightarrow PGL(m,\mathbb{C})$$

is called the <u>projective monodromy representation of</u> (3). Its image $\hat{G}$ is called the <u>projective monodromy group</u>. The Pfaffian system (3) is said to be <u>with finite projective monodromy</u> if $\hat{G}$ is a finite subgroup of $PGL(m,\mathbf{C})$. In this case, we have a finite Galois covering

$$\pi : X \longrightarrow M$$

which corresponds to ker $\hat{R}$. Put

$$\hat{Y} = \mathbb{P}^{s-1} = \mathbf{C}^s/\mathbf{C}^* \quad (s = m^2).$$

Then a fundamental solution F of (3) induces a holomorphic mapping

$$\hat{F}: X - \pi^{-1}(B) \longrightarrow \hat{Y}. \qquad \cdots\cdot(11)$$

(B is the polar set of Ω.)

The Pfaffian system (3) is said to be <u>of meromorphic type with finite projective monodromy</u> if (i) it has a finite projective monodromy group and (ii) $\hat{F}$ in (11) can be extended to a meromorphic mapping $\hat{F}: X \longrightarrow \hat{Y}$.

A similar argument to the proof of Proposition 2.1 shows

<u>Proposition 4.1.</u> Every Pfaffian system of Fuchsian type with finite projective monodromy is of meromorphic type with finite projective monodromy.

A meromorphic mapping

$$g: M \longrightarrow \hat{N} = \hat{Y}/\hat{G}$$

for a finite subgroup $\hat{G}$ of $PGL(m,\mathbf{C})$ is said to be <u>$\hat{G}$-primitive</u> if (i) there is a hypersurface B of M such that g is holomorphic on $M - B$ and $g(M-B)\cap\hat{\mu}(\hat{\Delta}) = \emptyset$, where

$$\hat{\Delta} = \left\{ \hat{y}\in\hat{Y} \;\middle|\; \det y = 0 \right\}$$

and $\hat{\mu}: \hat{Y} \longrightarrow \hat{N}$ is the natural projection ($\hat{y}$ is the image of y under the natural projection $\mathbb{C}^s \longrightarrow \hat{Y} = \mathbb{P}^{s-1}$) and (ii) a similar condition to (ii) in Definition 3.1.

The $(m \times m)$-matrix-valued meromorphic 1-form $\Xi = y^{-1}dy$ on $\mathbb{C}^s$ $(s = m^2)$ is $GL(m,\mathbb{C})$-invariant. In particular, it is $\mathbb{C}^*$-invariant. Hence it can be regarded as an $(m \times m)$-matrix-valued meromorphic 1-form on $\hat{Y} = \mathbb{P}^{s-1}$ and is $PGL(m,\mathbb{C})$-invariant.

A similar argument to the proof of Theorem 3.4 shows

Theorem 4.2. Let M be a complex manifold and $\hat{G}$ be a finite subgroup of $PGL(m,\mathbb{C})$. Then every Pfaffian system on M of meromorphic type with the projective monodromy group $\hat{G}$ can be obtained by Ω in (10) for a $\hat{G}$-primitive meromorphic mapping $f: M \longrightarrow \hat{N} = \mathbb{P}^{s-1}/\hat{G}$ $(s = m^2)$.

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